Topology and its Applications 155 (2008) 1493–1501

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Urysohn universal space, its development and Hausdorff's approach

Miroslav Hušek [a,b,*,1]

[a] J.E. Purkyně University, České mládeže 8, 400 96 Ústí nad Labem, Czech Republic
[b] Charles University, Sokolovská 83, 18675 Prague, Czech Republic

ARTICLE INFO

Article history:
Received 2 February 2007
Received in revised form 12 April 2007

MSC:
54E35
54G99
54-03

Keywords:
Separable metric space
Universal space

ABSTRACT

Three approaches to a direct construction of Urysohn universal space are compared, namely those of Urysohn, Hausdorff and Katětov. More details are devoted to the unpublished Hausdorff's approach that is shown to work in a more general situation, too.

We shall compare three constructions of Urysohn universal space, namely the original one by Urysohn, an unpublished one by Hausdorff, and finally a general construction given by Katětov. We show that the idea of the Hausdorff's construction works for higher cardinals too, like the Katětov's one—the main difference is that the former one gives a concrete description of the space, which may be useful in some situations.

We shall not deal with some abstract constructions implying existence of Urysohn universal space like Jónsson classes or Fraïssé limits.

1. Historical view

M. Fréchet defined metric spaces in 1905 (the term *metric spaces* comes from Hausdorff (1914)) and proved in [1] that every separable metric space can be isometrically embedded into l_∞. He repeated the result in [2]. At the end of the latter paper Fréchet asks if it is possible to have a separable space instead of l_∞. Before the paper was submitted (the date under the paper is August 21, 1924), he informed about that question Aleksandrov and Urysohn (probably during summer 1924).

In about the middle of July 1924, both Aleksandrov and Urysohn visited Hausdorff in Bonn. Then they spent one week by Brouwer and several days in Paris (without meeting any mathematician, they planned to stop by Fréchet on their way back). At the end of July they came to Bourg de Batz (now called Batz-sur-Mer) in Bretagne. On August 3 they sent a letter to Hausdorff where Urysohn announced a construction of a separable complete metric space containing isometrically any other separable metric space:

> … In dieser letzten Hinsicht ist es Urysohn gelungen einen (in Ihrem Sinne) vollständigen metrischen Raum mit abzählbarer dichter Teilmenge, der einen jeden anderen separablen metrischen Raum isometrisch enthält und ausser-

* Corresponding author at: Charles University, Sokolovská 83, 18675 Prague, Czech Republic.
E-mail addresses: husek@sci.ujep.cz, mhusek@karlin.mff.cuni.cz.
1 The author acknowledges supports of the grants MSM 0021620839 and GAČR 201/06/0018 of Czech Republic and of University J.E. Purkyně.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.03.020

> dem eine recht starke Homogenitätsbedingung füllt, zu konstruieren; letzterer besteht darin, dass man den ganzen Raum (isometrisch) so auf sich selbst abbilden kann, dass dabei eine beliebige endliche Menge M in eine ebenfalls beliebige, der Menge M kongruente Menge M_1 übergeführt wird. Es lässt sich noch beweisen, dass dieser Raum der einzige vollständige separable Raum ist, der diese beiden Eigenschaften (die Maximal- und die Homogenitätseigenschaft) besitzt; man dürfte ihn als den "Universellen metrischen separablen Raum" bezeichnen.

The letter does not contain any detail of a construction (and does not mention Fréchet). Also, it seems that such a question was not under discussion with Hausdorff any time before.

In Hausdorff's heritage, several pages of unpublished notes were found concerning his own look at universal metric separable spaces. Very probably, after reading the Urysohn's result he started to construct a universal space by his own.

On the first page dated August 9, 1924, he reproved the Fréchet's result on universality of l_∞ (without mentioning Fréchet). He notices that l_∞ is not separable and adds a remark that Urysohn constructed a separable universal space. The next page dated August 10, 1924, starts a construction of a universal metric separable space. Hausdorff proves that his construction gives a separable metric space containing isometrically all separable metric spaces and starts to prove homogeneity but did not finish it. On August 11, he writes in details about his procedure to Aleksandrov and Urysohn to Bourg de Batz (see [9]) and asks for details of Urysohn's construction. Although it is probable that Urysohn read the letter, no reply came back (Urysohn died on August 17).

The next notes from the Hausdorff's heritage have a comment *middle of August* (*Urysohn* † *17.8.1924*)—see [3], where these notes are published. They contain his previous construction (written in a more elegant way) and continue in proving the homogeneity of his space. The proof has not been finished, but the remaining part is not difficult to complete (see the next section).

The Urysohn universal space was announced in C. R. Acad. Paris in 1925 (séance 2.2.1925), [10]. The announcement contains a sketch of the proof and other properties of the space, namely universality of spheres and example of 4-point metric space that cannot be embedded isometrically into l_2. All the details were published in Bulletin Sci. Math. in 1927 (with the same title as the announcement); the paper was prepared by Aleksandrov. The Hausdorff's approach was not mentioned and Hausdorff himself has not published it (and, probably, never returned back to the problem).

After more than 60 years, M. Katětov presented at the Prague Topological Symposium in 1986 his own construction of universal metric spaces (not only separable ones).

For an infinite cardinal κ he defines a metric space X to be *κ-homogeneous* if every isometry between two subsets of X of cardinalities less than κ can be extended to an isometry of X. Urysohn's homogeneity coincides with Katětov's ω-homogeneity.

For an infinite cardinal κ he defines a metric space X to be *κ-universal* if every metric space of cardinality at most κ can be embedded isometrically into X. When every metric space having weight at most κ can be embedded isometrically into X, he calls X *strongly κ-universal.*

Some Katětov's results:

(1) *If $\kappa = \kappa^{<\kappa} > \omega$ there exists exactly one* (*up to isometry*) *strongly κ-universal κ-homogeneous metric space of weight κ and that space is complete.*
(2) *If $\kappa < \kappa^{<\kappa}$ then there exists no κ-universal κ-homogeneous space of weight κ.*
(3) *There exists a meager strongly ω-universal ω-homogeneous separable space.*

The last result answers Urysohn's question whether there exists a non-complete universal metric separable space that is ω-homogeneous. Urysohn's "universal" coincides with Katětov's "strongly ω-universal".

The Katětov's approach was used, e.g., by V.V. Uspenskii for other deep and interesting results about Urysohn universal space (see, e.g., [13,14] or [15] in this volume for other references).

2. Three constructions

2.1. Urysohn's construction

This construction is published and full details can be found in [11] (it was republished in [12]). Nevertheless, we shall briefly go through the construction because of completeness (and also because the original is not always accessible now).

Urysohn first constructs a countable space U_0 having for distances rational numbers and containing isometric copies of any other such space. The completion U of U_0 is a universal separable space. We shall use the original notation; sometimes, also in this issue, the space U_0 is denoted by $U_{\mathbb{Q}}$.

To define a convenient metric on $U_0 = \{a_1, a_2, a_3, \ldots\}$, he first orders finite subsets of positive rationals into a sequence $\{Q_n\}$ such that, except for the first member, the index n is larger than the cardinality of Q_n. Every Q_n has a fixed order of its elements (say, for the next definition of metric, $\rho_1, \rho_2, \ldots, \rho_q$). The metric is defined by induction, starting with $\rho(a_1, a_1) = 0$. If all the distances $\rho(a_i, a_k)$ are defined for $i, k \leqslant n$, the distance $\rho(a_{n+1}, a_j)$ equals either to ρ_j (the jth member of Q_n) or to $\max\{\rho(a_i, a_k);\ i, k \leqslant n\}$, depending whether the inequalities $|\rho_i - \rho_k| \leqslant \rho(a_i, a_k) \leqslant \rho_i + \rho_k$ hold for all $i, k \leqslant n$ or not. The triangle inequality for ρ is proved by induction according its definition.

The first case in the definition of the metric is important for the whole construction (the value of metric in the second case is defined so that the triangle inequality trivially holds). If one takes a segment $a_1, a_2, \dots, a_n$ of U_0 and positive rational numbers $\mu_1, \dots, \mu_n$ such that $|\mu_i - \mu_j| \leqslant \rho(a_i, a_j) \leqslant \mu_i + \mu_j$ and the point a_{m+1}, where $Q_m = \{\mu_1, \dots, \mu_n\}$ (recall that $m > n$) then $\rho(a_{m+1}, a_i) = \mu_i$ for all $i \leqslant n$.

Now, it suffices to realize (it is not so simple but not too difficult) that any finite set $a_{k_1}, \dots, a_{k_n}$ in U_0 and positive rational numbers $\mu_{k_1}, \dots, \mu_{k_n}$ such that $|\mu_{k_i} - \mu_{k_j}| \leqslant \rho(a_{k_i}, a_{k_j}) \leqslant \mu_{k_i} + \mu_{k_j}$ can be completed by "missing" numbers μ_p to get the whole segment $a_1, a_2, \dots, a_{k_n}$ satisfying the above inequalities.

It follows that U_0 *is a universal space for all countable metric spaces having rationals for values of their metrics*.

The next step is to extend the previous procedure to the completion U of U_0, which gives as a consequence that U *is universal for all separable metric spaces*. In fact, the following result is proved, denoted by Urysohn as Theorem I:

Theorem I. *For any finite subset $x_1, \dots, x_n$ of U and positive real numbers $\alpha_1, \dots, \alpha_n$ such that $|\alpha_i - \alpha_j| \leqslant \rho(x_i, x_j) \leqslant \alpha_i + \alpha_j$ one can find $y \in U$ such that $\rho(y, x_i) = \alpha_i$ for every $i = 1, \dots, n$.*

To prove that, Urysohn finds $a_1, \dots, a_n \in U_0$ and rationals $\nu_1, \dots, \nu_n$ close to corresponding given points x_i and numbers α_i, and satisfying corresponding inequalities. By the previous construction, there is a point $y_1 \in U_0$ with $\rho(y_m, a_i) = \nu_i$ and $|\rho(y_1, x_i) - \alpha_i| < \alpha/2 = \min\{\alpha_i\}/2$.

Now, Urysohn adds y_1 to the set $x_1, \dots, x_n$ and $\alpha/2$ to $\alpha_1, \dots, \alpha_n$ and repeats the procedure (with the approximating points a's and numbers ν's in U_0 closer to the corresponding points and numbers than before) to get a point $y_2 \in U_0$ with $|\rho(y_2, x_i) - \alpha_i| < \varepsilon/2$ and $\rho(y_2, y_1) < 3\alpha/4$.

The procedure results in a Cauchy sequence $\{y_m\}$ in U_0 converging to a requested point y.

A slightly stronger version of Theorem I (namely that the completion of a metric space with the approximation one-point extension property has the one-point extension property) was proved in [8].

2.2. Haudsorff's construction

The Hausdorff's construction has not been published (it will appear in collected work [3] of F. Hausdorff soon). So, we shall provide some details here; more details can be found in the last section of this contribution.

Hausdorff first defines a set $\mathcal{U}$ consisting of symmetric square matrices $(a_{i,j})_{i,j \leqslant n}$, $n \in \mathbb{N}$, with zeros in diagonal, where $a_{i,j} \geqslant 0$ and $a_{i,j} + a_{j,k} \geqslant a_{i,k}$ for all $i, j, k \leqslant n$. Thus one takes a finite pseudometric space, orders its points as $p_1, \dots, p_n$ and defines $a_{i,j}$ as the distance between p_i and p_j.

Denote the above matrix as α_n. For $m \leqslant n$ we denote by α_m the matrix generated in the above sense by $p_1, \dots, p_m$, i.e., a segment of α_n. The distance between such matrices is defined by the equality $d(\alpha_m, \alpha_n) = d(\alpha_n, \alpha_m) = a_{m,n}$. It is trivial that this distance satisfies triangle inequality on all segments of a given matrix from $\mathcal{U}$.

The function d can be extended to a distance between any elements of $\mathcal{U}$ by induction:

Definition 1. Let $\alpha_n, \beta_m \in \mathcal{U}$ and all the distances $d(\alpha_n, \beta_l)$, $d(\alpha_k, \beta_m)$ were defined for $k < n$, $l < m$. Then we define

$$d(\alpha_n, \beta_m) = d(\beta_m, \alpha_n) = \max\bigl\{\bigl|d(\alpha_n, \alpha_k) - d(\beta_m, \alpha_k)\bigr|, \bigl|d(\alpha_n, \beta_l) - d(\beta_m, \beta_l)\bigr|;\ k < n,\ l < m\bigr\}.$$

Notice that if β_m is a segment α_m of α_n, then this last definition of $d(\alpha_n, \beta_m)$ coincides with the former one.

Proposition 2. *$(\mathcal{U}, d)$ is a pseudometric space.*

Proof. The proof of triangle inequality $d(\alpha_n, \beta_m) + d(\beta_m, \gamma_p) \geqslant d(\alpha_n, \gamma_p)$ goes by induction on $n + m + p$ (it is trivially true for $m = n = p = 1$). Suppose that $d(\alpha_n, \gamma_p) = |d(\alpha_n, \alpha_k) - d(\gamma_p, \alpha_k)|$ for some $k < n$. Then

$$d(\alpha_n, \gamma_p) \leqslant \bigl|d(\alpha_n, \alpha_k) - d(\beta_m, \alpha_k)\bigr| + \bigl|d(\beta_m, \alpha_k) - d(\gamma_p, \alpha_k)\bigr| \leqslant d(\alpha_n, \beta_m) + d(\beta_m, \gamma_p). \qquad \square$$

Hausdorff identifies matrices having distance zero and denotes the resulting metric space of equivalent classes also by $\mathcal{U}$.

Proposition 3. *The metric space $\mathcal{U}$ is universal for all countable metric spaces.*

Proof. If $p_1, p_2, \dots$ is a countable metric space X, then the distance–matrices α_n of the segments $p_1, p_2, \dots, p_n$ form a subspace of $\mathcal{U}$ isometric to X. $\square$

It follows easily by induction (on $n + m$) that $d(\alpha_n, \beta_m)$ is a continuous function of numbers in the matrices. Thus matrices composed of rational numbers form a dense set in $\mathcal{U}$ (that assertion is just stated in the notes but, by my opinion, it needs a little work).

Proposition 4. *$\mathcal{U}$ is separable.*

As a consequence of previous assertions we have

Theorem 5. *The completion $\mathcal{V}$ of $\mathcal{U}$ is a separable universal space for all separable metric spaces.*

2.3. Katětov's construction

This construction is published in [5] which is not widely available. Nevertheless, it is described in the paper by V.V. Uspenskii in this volume and so, we shall mention an idea only.

Every nonempty metric space (S,d) determines a metric space $E(S)=\{f:S\to\mathbb{R};\ |f(p)-f(q)|\leqslant d(p,q)\leqslant f(p)+f(q)$ for every $p,q\in S\}$ endowed with the sup-metric. When one identifies $p\in S$ with the function $f(x)=d(x,p)$, one may assume $S\subset E(S)$.

If T is a subspace of S, then $E(T)$ embeds isometrically into $E(S)$: every $f\in E(T)$ extends to $f_S\in E(S)$, $f_S(s)=\inf\{d(s,t)+f(t); t\in T\}$. Thus we may consider $E(T)$ as a subspace of $E(S)$ and define for cardinals $\kappa>1$,

$$E(S,\kappa)=\bigcup\big\{E(T);\ T\subset S,\ 0<|T|<\kappa\big\}\subset E(S)$$

and by induction, starting with $S_0=S$,

$$S_{\alpha+1}=E\big(S_\alpha,\omega\cdot|\alpha|^+\big),\qquad S_\alpha=\bigcup_{\beta<\alpha}S_\beta,$$

the last case for limit α. Notice, that $|S_\alpha|\leqslant 2^{<\alpha}$.

If $\kappa=\kappa^{<\kappa}$ then, for any metric space S with $|S|\leqslant 2^\omega$, S_κ is a strongly κ-universal space from the first of the Katětov's results quoted above. For $\kappa=\omega$ we obtain a Urysohn universal space.

3. Homogeneity

When we said that there were no other contributions to a construction of a universal space between 1924 and 1986, it does not mean there were no papers dealing with properties of that space. Most of them concern homogeneity. Urysohn universal space is ω-homogeneous (according to the Katětov's terminology). Urysohn also showed that there are infinite isometric subsets of his universal space such that the isometry cannot be extended to the whole space. The cardinality of those subsets is 2^ω (one subset is a sphere together with its center, the other is a subset of that sphere; now, the complement (in U) of the first set is not connected, the complement of the other is connected).

In 1953, Mrówka published an example of countable isometric subsets of Urysohn universal space such that the isometry cannot be extended to an isometry of the whole space, [7].

Two years later, Huhunaišvili noted that the Urysohn's example with uncountable sets M,N implies a countable case: it suffices to take a countable dense subset A of M and its isometric image B in N, [4].

Huhunaišvili asks whether the isometry between *closed* countable subsets can be extended to the whole space. He answers the question in the negative using the Urysohn's approach:

Take a closed infinite ε-net $\{y_n\}$ in the sphere $S(a,1)$, where $\varepsilon<1$. The set $A=\{a\}\cup\{y_n\}$ is isometric to a subset B of $S(a,1)$. This isometry cannot be extended to the whole space.

If one goes even further and requests the original isometry to be between compact countable sets, the answer is now in the positive. In fact, the following result is proved in [4]: *Any isometry between totally bounded subsets of the Urysohn complete separable metric space can be extended to an isometry of the whole space.*

For more examples of isometries non-extendible to the whole Urysohn space, see Melleray's paper [6] in this volume.

3.1. Homogeneity of the Urysohn's universal space

To extend an isometry $a_i\rightsquigarrow b_i$ between two finite subsets $a_1,\dots,a_n$ and $b_1,\dots,b_n$ of U to an isometry on U, it suffices to extend isometrically both finite sets by a countable dense set $\{d_{n+1},d_{n+2},\dots\}$. Using Theorem I for $x_i=a_i$ and $\alpha_i=d(d_{n+1},a_i)$ one gets $y_{n+1}\in U$ such that $\{a_i,\dots,a_n,d_{n+1}\}$ is isometric to $\{b_1,\dots,b_n,y_{n+1}\}$. Repeating the same process for the new set $\{b_1,\dots,b_n,y_{n+1}\}$ as the set of x_i's, and distances of d_{n+1} to the points of this new set as α_i's, one gets a point z_{n+2} such that the sets $\{a_i,\dots,a_n,d_{n+1},z_{n+2}\}$ and $\{b_1,\dots,b_n,y_{n+1},d_{n+1}\}$ are isometric. This procedure gives two isometric dense sequences

$$\{a_1,\dots,a_n,d_{n+1},z_{n+2},d_{n+2},z_{n+3},\dots\}\sim\{b_1,\dots,b_n,y_{n+1},d_{n+1},y_{n+2},d_{n+2},\dots\}.$$

Because of completeness of U, this final isometry extends to an isometry of $\mathcal{U}$ onto itself.

The uniqueness of U announced in the letter to Hausdorff is an easy consequence of homogeneity. If P is another universal separable ω-homogeneous metric space, then the same procedure described in the previous paragraph gives an isometry between U and P: one starts with empty sets $\{a_i\}$, $\{b_i\}$ and countable dense sets $\{d_n\}$, $\{d'_n\}$ in U or P respectively.

3.2. Homogeneity of the Hausdorff's universal space

The proof of universality of the Hausdorff's space $\mathcal{V}$ was very simple. Urysohn used for his proof of universality of U more complicated Theorem I. But having Theorem I, Urysohn had a simple way to show homogeneity of his space, meanwhile Hausdorff had to prove a kind of Theorem I anyway.

We shall follow Hausdorff's notations. By indexed Greek letters like α_n we mean a member of $\mathcal{U}$ being a matrix $(n \times n)$ (then α_i, $0 < i < n$, is the ith segment of α_n, i.e., the left upper $(i \times i)$-submatrix of α_n). The lower Latin letters (indexed or not, like a, b_j) denote elements of $\mathcal{U}$ without specification of size of matrices.

For two finite or countable sequences in $\mathcal{U}$ having the same cardinality, Hausdorff uses the following notation:

$$\{a_1, \ldots, a_n\} \backsim \{b_1, \ldots, b_n\}$$

means isometry preserving indices, i.e., $d(a_i, a_j) = d(b_i, b_i)$, $i, j = 1, \ldots, n$.

Hausdorff first shows the following fact that follows directly from the definition of d:

Lemma 6. *Let $\alpha_n, \beta_m \in \mathcal{U}$ such that*

- *For every $j < m$ there is $i < n$ with $d(\beta_j, \alpha_i) = 0$;*
- $\{\alpha_1, \ldots, \alpha_{n-1}, \alpha_n\} \sim \{\alpha_1, \ldots, \alpha_{n-1}, \beta_m\}$.

Then $d(\alpha_n, \beta_m) = 0$.

So, if $\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_p, \ldots\} \sim \{\xi_1, \xi_2, \ldots\}$, then every ith member of the left sequence has zero distance from the ith member of the right sequence. Consequently, every sequence (finite or countable) $\{a_n\}$ in $\mathcal{U}$ is a subsequence of some sequence $\{\xi_1, \xi_2, \ldots\}$. (Recall that similar fact was used by Urysohn, too.)

Hausdorff now uses a very nice "tricky" way to prove the following analogy of Urysohn's Theorem I:

Proposition 7. *If $\{a_1, \ldots, a_m\} \sim \{\xi_1, \ldots, \xi_m\}$ and ξ_m is a segment of ξ_{m+1}, then there is some $a_{m+1} \in \mathcal{U}$ with $\{a_1, \ldots, a_{m+1}\} \sim \{\xi_1, \ldots, \xi_{m+1}\}$.*

Proof. Hausdorff shows the proof for $m = 3$, i.e., to the equivalence $\{\alpha_m, \beta_n, \gamma_p\} \sim \{\xi_1, \xi_2, \xi_3\}$ he constructs δ_q so that $\{\alpha_m, \beta_n, \gamma_p, \delta_q\} \sim \{\xi_1, \xi_2, \xi_3, \xi_4\}$.

There is some $\eta_{m+n+p} \in \mathcal{U}$ such that

$$\{\alpha_m, \beta_n, \gamma_p, \alpha_1, \ldots, \alpha_{m-1}, \beta_1, \ldots, \beta_{n-1}, \gamma_1, \ldots, \gamma_{p-1}\} \sim \{\eta_1, \ldots, \eta_{m+n+p}\}.$$

(We may assume $\eta_i = \xi_i$ for $i = 1, 2, 3$.) Then there is some $\zeta_{m+n+p+1}$ with

$$\{\eta_4, \ldots, \eta_{m+2}, \eta_1, \eta_{m+3}, \ldots, \eta_{m+n+1}, \eta_2, \eta_{m+n+2}, \ldots, \eta_{m+n+p}, \eta_3, \xi_4\} \sim \{\zeta_1, \ldots, \zeta_{m+n+p+1}\}.$$

The requested δ_q equals to $\zeta_{m+n+p+1}$. Indeed, by the first lemma one has $d(\alpha_m, \xi_m) = 0$, $d(\beta_n, \xi_{m+n}) = 0$, $d(\gamma_p, \xi_{m+n+p}) = 0$ so that

$$\{\alpha_m, \beta_n, \gamma_p, \delta_q\} \sim \{\eta_1, \eta_2, \eta_3, \eta_4\} = \{\xi_1, \xi_2, \xi_3, \xi_4\}. \qquad \square$$

Corollary 8. *If $\{a_1, \ldots, a_n\} \sim \{b_1, \ldots, b_n\}$ in $\mathcal{U}$ and $b_{n+1} \in \mathcal{U}$ there exists an $a_{n+1} \in \mathcal{U}$ such that $\{a_1, \ldots, a_{n+1}\} \sim \{b_1, \ldots, b_{n+1}\}$.*

The following weaker form of Theorem I can now be proved in the same way as Theorem I.

Proposition 9. *If $\{a_1, \ldots, a_n\} \sim \{b_1, \ldots, b_n\}$ in $\mathcal{U}$ then there exists an isometry of $\mathcal{V}$ onto itself extending that given isometry.*

Here the Hausdorff's notes finish with the remark that one should now extend the preceding result to finite subsets $\{a_1, \ldots, a_n\} \sim \{b_1, \ldots, b_n\}$ in $\mathcal{V}$. We do not know why Hausdorff did not finish his proof (at least, no continuation was found). It seems probable that he knew how to finish the proof since remaining procedures are quite natural. We can suggest one of them.

To use the same technique as in the proof of Proposition 9 one needs the assertion of Corollary 8 for $\mathcal{V}$ instead of for $\mathcal{U}$, in fact a little less. Since every finite metric space can be isometrically embedded in $\mathcal{U}$, it suffices to prove the assertion of Corollary 8 for $\{a_i\} \subset \mathcal{V}$, $\{b_i\} \subset \mathcal{U}$ and $b_{n+1} \in \mathcal{U}$. To simplify our procedure, we shall first prove the following lemma.

Lemma 10. *For every finite set $\{a_1, \ldots, a_n\} \subset \mathcal{V}$ there exist sequences $\{a_{k,i}\}_k$ in $\mathcal{U}$ converging to a_i such that $d(a_{k,i}, a_{k,j}) = d(a_i, a_j)$ for all possible i, j, k.*

Proof. We shall assume that $d(a_i, a_j) \neq 0$ for $i \neq j$ and denote $r = \min\{d(a_i, a_j); i \neq j\}$. Let $\{a'_{k,i}\}_k$ be sequences from $\mathcal{U}$ converging to a_i. We may assume that $|d(a'_{k,i}, a'_{k,j}) - d(a_i, a_j)| \leqslant r2^{-k}$ for all i, j, k.

Let $a_{k,1} = a'_{k,1}$ for all $k = 1, 2, \ldots$. Suppose now that $1 < m \leqslant n$ and sequences $\{a_{k,i}\}_k$ having the requested properties are constructed for all $i < m$. Moreover, we shall assume that $d(a_{k,i}, a'_{k,i}) \leqslant r2^{-k-n+i}$ (true for $i = 1$). We shall now construct a sequence $\{a_{k,m}\}_k$ having similar corresponding properties.

Proof goes by induction on indices k. For $k = 1$ and $i < m$ let $\eta_i = d(a_m, a_i)$, $\eta_m = r2^{m-n-1}$. Then by Corollary 8 there is a point $a_{1,m}$ with $\eta_i = d(a_{1,m}, a_{1,i})$ for $i < m$ and $\eta_m = d(a_{1,m}, a'_{1,m})$. To use Corollary 8 one must check that

$$|\eta_i - \eta_j| \leqslant d(a_{1,i}, a_{1,j}) \leqslant \eta_i + \eta_j \quad \text{for } i, j < m,$$
$$|\eta_i - \eta_m| \leqslant d\big(a_{1,i}, a'_{1,m}\big) \leqslant \eta_i + \eta_m \quad \text{for } i < m.$$

The first row is clear. The second one follows from the following inequalities:

$$d\big(a'_{1,i}, a'_{1,m}\big) - d\big(a_{1,i}, a'_{1,i}\big) \leqslant d\big(a_{1,i}, a'_{1,m}\big) \leqslant d\big(a_{1,i}, a'_{1,i}\big) + d\big(a'_{1,i}, a'_{1,m}\big),$$
$$d(a_i, a_m) - r/2 \leqslant d\big(a'_{1,i}, a'_{1,m}\big) \leqslant d(a_i, a_m) + r/2$$

(we must realize that $r(2^{-1-n+i} + 1/2) \leqslant r2^{-1-n+m}$).

If $a_{k,m}$ with the requested properties are constructed for all $k < p$ one can construct $a_{p,m}$ in the same way using $\eta_i = d(a_m, a_i)$, $\eta_m = r2^{m-n-p}$ and the points $\{a_{p,i}\}_{i<m}$ and $a'_{p,m}$. □

It remains to prove the following assertion.

Proposition 11. *Let $\{a_1, \ldots, a_n\} \sim \{b_1, \ldots, b_n\}$ in $\mathcal{V}$ and suppose the latter set to be a part of $\mathcal{U}$. Then for any $b_{n+1} \in \mathcal{U}$ there exists some $a_{n+1} \in \mathcal{V}$ such that $\{a_1, \ldots, a_{n+1}\} \sim \{b_1, \ldots, b_{n+1}\}$.*

Proof. For every a_i we fix a sequence $\{a_{k,i}\}_k$ in $\mathcal{U}$ converging to a_i such that for all possible i, j, k we have $d(a_{k,i}, a_{k+1,i}) \leqslant 2^{-k}$ and $d(a_{k,i}, a_{k,j}) = d(a_i, a_j)$ (by the previous corollary). Let φ be an isometric embedding of $\{a_i\}_i \cup \{a_{k,i}\}_{k,i}$ onto a set $\{c_i\}_i \cup \{c_{k,i}\}_{k,i} \subset \mathcal{U}$. There is some $c_{i+1} \in \mathcal{U}$ such that $\{b_i\}_{i=1}^{n+1}$ is pointwise isometric to $\{c_i\}_{i=1}^{n+1}$.

Using the same procedure as in the previous lemma, one can construct a sequence $\{a_{k,n+1}\}_k$ in $\mathcal{U}$ such that for $i = 1, \ldots, n$ and all k

$$d(a_{k,n+1}, a_{k,i}) = d(b_{n+1}, b_i), \qquad d(a_{k,n+1}, a_{k+1,n+1}) \leqslant 2^{-k}.$$

Now, $\{a_{k,n+1}\}_k$ is a Cauchy sequence and, thus, converges to a point a_{n+1} in $\mathcal{V}$. Clearly, $\{a_1, \ldots, a_{n+1}\} \sim \{b_1, \ldots, b_{n+1}\}$. □

3.3. Homogeneity of the Katětov's universal space

Katětov first proves that every isometry of S extends uniquely to an isometry of $E(S, \alpha)$ and then uses the standard procedure (used also by Urysohn and Hausdorff) of extending an isometry to sets containing dense subsets.

To prove uniqueness of his universal spaces, Katětov uses the original Urysohn procedure modified for uncountable densities.

4. Nonseparable spaces

We shall now modify the Hausdorff's construction to higher cardinals to see which parts of the Katětov's results can be proved by this approach.

In the sequel, the ordinal number 0 is excluded from our indexing systems (it would cause difficulties in using sums of ordinals). Nevertheless, we shall not always write $\xi > 0$ because it would make some indices long.

4.1. Construction

For an ordinal $\xi > 0$ denote by A_ξ a symmetric square matrix $(a_{\alpha,\beta})_{0<\alpha,\beta\leqslant\xi}$ having zeros in diagonal and such that

$$a_{\alpha,\beta} \leqslant a_{\alpha,\gamma} + a_{\gamma,\beta}$$

for any $\alpha, \beta, \gamma \leqslant \xi$. Such a matrix is a distance matrix $(d(p_\alpha, p_\beta))_{\alpha,\beta\leqslant\xi}$ of a sequence $\{p_\alpha\}_{\alpha\leqslant\xi}$ of points in a pseudometric space.

For $\zeta < \xi$, a segment A_ζ of A_ξ is a left upper $(\zeta \times \zeta)$-submatrix of A_ξ (in the previous representation, A_ζ corresponds to the distance matrix of the initial part $\{p_\alpha\}_{\alpha\leqslant\zeta}$ of $\{p_\alpha\}_{\alpha\leqslant\xi}$).

Denote by $\mathcal{U}$ the class of all such matrices A_ξ for all ordinals $\xi > 0$. The elements of $\mathcal{U}$ are denoted as A_ξ, B_τ, C_ζ, where the indices mean the size of matrices. In case of two matrices with the same capital, say A_ξ, A_ζ, one of the matrix is a segment of the other.

If $A_\xi \in \mathcal{U}$, $0 < \zeta \leqslant \eta \leqslant \xi$, the function $d(A_\zeta, A_\eta) = d(A_\eta, A_\zeta) = \alpha_{\zeta,\eta}$ is a pseudometric on the set of segments of A_ξ.

As in the finite case, one can now define distances between any two elements of $\mathcal{U}$ by induction.

Definition 12. Define a symmetric function d on $\mathcal{U} \times \mathcal{U}$ by induction:

Let $d(A_1, A_1) = 0$. If $d(A_\varphi, B_\psi)$ is defined whenever $0 < \phi \leqslant \xi, 0 < \psi \leqslant \zeta$ where at least one of the inequalities $\leqslant$ is not equality, then

$$d(A_\xi, B_\zeta) = d(B_\zeta, A_\xi) = \sup\bigl\{\bigl|d(A_\xi, A_\mu) - d(A_\mu, B_\zeta)\bigr|, \bigl|d(A_\xi, B_\nu) - d(B_\nu, B_\zeta)\bigr|; \mu < \xi, \nu < \zeta\bigr\}.$$

To prove that d is a pseudometric on the class $\mathcal{U}$, we must show that it is finite and satisfies triangle inequality $d(A_\xi, B_\zeta) \leqslant d(A_\xi, C_\tau) + d(C_\tau, B_\zeta)$.

If B_ζ coincides with a segment of A_ξ then the previous definition of d coincides the former one. Thus the above triangle inequality holds if A_ξ, B_ζ, C_τ are segments of some element of $\mathcal{U}$. Also the following special case of triangle inequality holds (substitute $\mu = 1$ into the definition of d):

$$d(A_\xi, A_0) \leqslant d(A_\xi, B_\zeta) + d(B_\zeta, A_0).$$

Lemma 13. *The function d is finite, i.e., has its values in $\mathbb{R}$.*

Proof. We shall prove by induction that $d(A_\xi, B_\zeta) \leqslant a_{1,\xi} + b_{1,\zeta}$. The inequality is true for $\xi = \zeta = 1$. Assume $d(A_\mu, B_\nu) \leqslant a_{1,\mu} + b_{1,\nu}$ for $0 < \mu \leqslant \xi$ and $0 < \nu \leqslant \zeta$, where at least one inequality $\leqslant$ is not equality. Consider $|d(A_\xi, A_\mu) - d(A_\mu, B_\zeta)|$ (the procedure for the other absolute value in the definition of d is similar).

Suppose first that $|d(A_\xi, A_\mu) - d(A_\mu, B_\zeta)| = d(A_\xi, A_\mu) - d(A_\mu, B_\zeta)$. We have

$$d(A_\xi, A_\mu) - d(A_\mu, B_\zeta) \leqslant a_{\xi\mu} - (a_{1,\mu} - b_{1,\zeta}) \leqslant a_{1,\xi} + b_{1,\zeta}.$$

In the other case we have

$$d(A_\mu, B_\zeta) - d(A_\xi, A_\mu) \leqslant (a_{1,\mu} + b_{1,\zeta}) - a_{\xi,\mu} \leqslant a_{1,\xi} + b_{1,\zeta}.$$

Therefor all the numbers in the right-hand side of the definition of d are bounded by $a_{1,\xi} + b_{1,\zeta}$. □

Proposition 14. *The triangle inequality $d(A_\xi, B_\zeta) \leqslant d(A_\xi, C_\tau) + d(C_\tau, B_\zeta)$ holds for any choice of A_ξ, B_ζ, C_τ from $\mathcal{U}$.*

Proof. The assertion is clearly valid for $\xi = \zeta = \tau = 1$. Take now arbitrary ξ, ζ, τ and assume that the triangle inequality is valid for any $0 < \xi' \leqslant \xi$, $0 < \zeta' \leqslant \zeta$, $0 < \tau' \leqslant \tau$ where at least one of the inequalities $\leqslant$ is not equality.

If $d(A_\xi, B_\zeta) > d(A_\xi, C_\tau) + d(C_\tau, B_\zeta)$ then there exists $\mu < \xi$ (or $\nu < \zeta$) such that $|d(A_\xi, A_\mu) - d(A_\mu, B_\zeta)| > d(A_\xi, C_\tau) + d(C_\tau, B_\zeta)$ (or $|d(A_\xi, B_\nu) - d(B_\nu, B_\zeta)| > d(A_\xi, C_\tau) + d(C_\tau, B_\zeta)$, respectively). But

$$\bigl|d(A_\xi, A_\mu) - d(A_\mu, B_\zeta)\bigr| \leqslant \bigl|d(A_\xi, A_\mu) - d(A_\mu, C_\tau)\bigr| + \bigl|d(A_\mu, C_\tau) - d(A_\mu, B_\zeta)\bigr|$$

and the first absolute value on the right-hand side is at most $d(A_\xi, C_\tau)$ by definition of d and the second one is at most $d(C_\tau, B_\zeta)$ by our inductive assumption. Similarly for the other possibility. We have got a contradiction. □

Definition 15. For an infinite cardinal κ denote by $\mathcal{U}_\kappa$ the subset of $\mathcal{U}$ composed of matrices having size less than κ.

The construction gives directly universality:

Proposition 16. *The space $\mathcal{U}_\kappa$ is a κ-universal space.*

Proof. Take a pseudometric space P of cardinality κ and well-order its elements as $\{p_\alpha\}_{\alpha<\kappa}$. For every $\xi < \kappa$ let A_ξ be the distance matrix of the sequence $\{p_\alpha\}_{\alpha\leqslant\xi}$. Then the subset $\{A_\xi; \xi < \kappa\}$ of $\mathcal{U}_\kappa$ is isometric to P. □

4.2. Homogeneity

The next assertion is quite easy:

Lemma 17. *If φ is an isometry of a sequence $\{A_\alpha\}_{\alpha<\zeta}$ into $\mathcal{U}$ then for any $B \in \mathcal{U}$ there exists an extension A_ζ of the matrices $\{A_\alpha\}_{\alpha<\zeta}$ such that the extension of φ, mapping A_ζ into B, is an isometry.*

We must prove a similar result with switched existence of B, A_ζ. The procedure will modify the Hausdorff's one for $\kappa = \omega$. We shall say that two families $\mathcal{A} = \{A^\alpha_{\beta_\alpha}\}_{\alpha<\xi}$ and $\mathcal{B} = \{B^\alpha_{\gamma_\alpha}\}_{\alpha<\xi}$ in $\mathcal{U}$ are *pointwise isometric* if the mapping $\varphi : \mathcal{A} \to \mathcal{B}$ assigning $B^\alpha_{\gamma_\alpha}$ to $A^\alpha_{\beta_\alpha}$ is isometric.

Lemma 18. *Let $A_\zeta, B_\mu \in \mathcal{U}$ satisfying*

(1) *$d(B_\mu, A_\alpha) = d(A_\zeta, A_\alpha)$ for every $\alpha < \zeta$;*
(2) *for every $\beta < \mu$ there is some $\alpha_\beta < \zeta$ with $d(B_\beta, A_{\alpha_\beta}) = 0$.*

Then $d(B_\mu, A_\zeta) = 0$.

Proof. By definition,

$$d(B_\mu, A_\zeta) = \sup_{\alpha<\zeta,\beta<\mu} \bigl\{\bigl|d(B_\mu, A_\alpha) - d(A_\alpha, A_\zeta)\bigr|, \bigl|d(B_\mu, B_\beta) - d(B_\beta, A_\zeta)\bigr|\bigr\}.$$

The first absolute value in the supremum equals to 0 according our condition (1). Using the condition (2), one may substitute A_{α_β} instead of B_β in the second absolute value and get 0 again by (1). □

In the sequel, $\sum'_{\alpha<\zeta} \beta_\alpha$ denotes the usual sum of ordinals with the exception that the number 0 is not used, i.e., it is the set $\{(\alpha, \gamma);\ 0 < \alpha < \zeta,\ 0 < \gamma < \beta_\alpha\}$ endowed with the lexicographic order.

Corollary 19. *If $\{B^\alpha_{\beta_\alpha}\}_{\alpha<\zeta} \subset \mathcal{U}$ and $\{\{B^\alpha_\gamma\}_{\gamma\leqslant\beta_\alpha}\}_{\alpha<\zeta}$ is pointwise isometric to $\{A_\delta\}_{\delta<\sum'_{\alpha<\zeta}\beta_\alpha}$ then $d(B^\alpha_\gamma, A_{(\alpha,\gamma)}) = 0$ for every $0 < \alpha < \zeta$, $0 < \gamma \leqslant \beta_\alpha$.*

Proof. Proof goes by induction. The equality $d(B^1_1, A_1) = 0$ trivially holds. Suppose $d(B^\alpha_\gamma, A_{(\alpha,\gamma)}) = 0$ for every $(\alpha, \gamma) < (\alpha', \gamma')$. To get the equality $d(B^{\alpha'}_{\gamma'}, A_{(\alpha',\gamma')}) = 0$ we shall use Lemma 18. Its conditions are satisfied according the assumption in the assertion and our inductive hypothesis. Indeed, e.g. for condition (1) we have $d(B^{\alpha'}_{\gamma'}, A_{(\alpha,\gamma)}) = d(B^{\alpha'}_{\gamma'}, B^\alpha_\gamma) = d(A_{(\alpha',\gamma')}, A_{(\alpha,\gamma)})$. □

We shall now modify the elegant step of the Hausdorff's procedure to show κ-homogeneity of $\mathcal{U}_\kappa$.

Lemma 20. *Let $A_\lambda \in \mathcal{U}$. Then any isometry of $\{A_\alpha;\ 0 < \alpha < \lambda\}$ into $\mathcal{U}$ can be extended to an isometry of $\{A_\alpha;\ 0 < \alpha \leqslant \lambda\}$ into $\mathcal{U}$.*

Proof. Let the isometry be of the form $\varphi = \{A_\alpha \rightsquigarrow B^\alpha_{\beta_\alpha}\}$ for $0 < \alpha < \lambda$. There is some $C \in \mathcal{U}$ that is pointwise isometric to $\{A_\delta\}_{\delta<\lambda} \cup \{\{B^\alpha_\gamma\}_{\gamma<\beta_\alpha}\}_{\alpha<\lambda}$. We may assume $C_\delta = A_\delta$ for $0 < \delta < \lambda$. Let p be a permutation of $\lambda + \sum'_{\alpha<\lambda} \beta_\alpha$ defined by

$$p(\alpha) = (\alpha, \beta_\alpha) \quad \text{for } \alpha < \lambda, \qquad p(\alpha, \gamma) = (\alpha, \gamma) \quad \text{otherwise}.$$

The permutation p maps $\lambda + \sum'_{\alpha<\lambda} \beta_\alpha$ onto $\sum'_{\alpha<\lambda}(\beta_\alpha + 1)$. There is $\{D_\alpha;\ 0 < \alpha \leqslant \tau = \sum'_{\alpha<\lambda}(\beta_\alpha + 1) + 1\}$ pointwise isometric to $\{C_{p(\delta)}\}_{p(\delta)\leqslant\sum'_{\alpha<\lambda}(\beta_\alpha+1)} \cup \{A_\lambda\}$. According to Corollary 19 we have $d(D_{(\alpha,\gamma)}, B^\alpha_\gamma) = 0$ and $d(D_\tau, D_{(\alpha,\gamma)}) = d(A_\lambda, C_{(\alpha,\gamma)})$ for all possible indices.

Consequently, for every $\alpha < \lambda$ we have

$$d\bigl(D_\tau, B^\alpha_{\beta_\alpha}\bigr) = d(D_\tau, D_{(\alpha,\beta_\alpha)}) = d(A_\lambda, C_{(\alpha,\beta_\alpha)}) = d(A_\lambda, A_\alpha).$$

The requested extension of the isometry φ maps A_λ to D_τ. □

The result is valid if one substitutes $\mathcal{U}_\kappa$ for $\mathcal{U}$ with an additional assumption, namely that κ is regular.

The previous lemma has the following consequence (the idea of the proof is the same as that in Katětov's proof for his classes).

Lemma 21. *If $\kappa > \omega$ then $\mathcal{U}_\kappa$ is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\mathcal{U}_\kappa$ and x be its limit in a completion of $\mathcal{U}_\kappa$. Then the space $\{x_n\} \cup \{x\}$ can be isometrically embedded onto some $\{A_\alpha\}_{\alpha\leqslant\omega} \subset \mathcal{U}_\kappa$, where A_n corresponds to x_n. The isometry $A_n \rightsquigarrow x_n$ between $\{A_\alpha\}_{\alpha<\omega}$ and $\{x_n\}$ extends onto $\{A_\alpha\}_{\alpha\leqslant\omega}$. Clearly, the image of A_ω under that isometry gives a limit of $\{x_n\}$ in $\mathcal{U}_\kappa$ (and equals to x). □

Since isometries into complete spaces extend to isometries of closures, lemma gives immediately

Theorem 22. *If $\kappa > \omega$ then $\mathcal{U}_\kappa$ is a strongly κ-universal space.*

Notice that we claim nothing about weight of $\mathcal{U}_\kappa$.

Using the previous crucial Lemma 19 in the same way as was used by Urysohn and Hausdorff, we can now prove the following result (regularity of κ is needed according the note following Lemma 20).

Proposition 23. *Let κ be regular, A, B be isometric subsets of $\mathcal{U}_\kappa$ of cardinality $\lambda < \kappa$ and $D \subset \mathcal{U}_\kappa$ have cardinality at most κ. Then the isometry can be extended to some sets P, Q in $\mathcal{U}_\kappa$ with $P \supset A \cup D$, $Q \supset B \cup D$. For $\kappa > \omega$ one can construct such P, Q closed.*

Proof. Let $D = \{d_\alpha;\ 0 < \alpha < \kappa\}$, φ_A (or φ_B) is an isometry of A (or B) onto $\{A_\alpha\}_{\alpha<\lambda}$ (or onto $\{B_\alpha\}_{\alpha<\lambda}$, respectively).

By Lemma 17, φ_A extends to $A \cup \{d_1\}$ onto $\{A_\alpha\}_{\alpha<\lambda+1}$ and the pointwise isometry between $\{A_\alpha\}_{\alpha<\lambda}$ and $\{B_\alpha\}_{\alpha<\lambda}$ extends to $\{A_\alpha\}_{\alpha<\lambda+1}$ onto $\{B_\alpha\}_{\alpha<\lambda+1}$. Now, using Lemma 19 we can extend the isometry φ_B of $\{B_\alpha\}_{\alpha<\lambda+1}$ onto $B \cup \{x_1\}$ for some element x_1 of $\mathcal{U}_\kappa$. Thus we have an isometry between $A \cup \{d_1\}$ and $B \cup \{x_1\}$.

Repeating the same procedure starting with $B \cup \{x_1, d_1\}$, we get an isometry between $B \cup \{x_1, d_1\}$ and $A \cup \{d_1, y_1\}$ for some point y_1 of $\mathcal{U}_\kappa$. By induction one constructs an isometry between the sets $A \cup \{d_1, y_1, d_2, y_2, \ldots, d_\alpha, y_\alpha, \ldots\}_{\alpha<\kappa}$ and $B \cup \{x_1, d_1, x_2, d_2, \ldots, x_\alpha, d_\alpha, \ldots\}_{\alpha<\kappa}$. Those sets are the requested sets P, Q. If $\kappa > \omega$, then the closures of P and Q are complete and, therefore, the isometry extends to the closures. □

As a consequence, we have (recall that $\kappa = \kappa^{<\kappa}$ iff κ is regular and $(2^\omega)^{<\kappa}$).

Theorem 24. *If $\kappa = \kappa^{<\kappa} > \omega$ then $\mathcal{U}_\kappa$ is a strongly κ-universal and κ-homogeneous space having cardinality (and, thus, its weight) equal to κ.*

Proof. The cardinality of $\mathcal{U}_\kappa$ is at most $(2^\omega)^{<\kappa}$, which is κ under our assumption $\kappa = \kappa^{<\kappa}$. Proposition 23 gives κ-homogeneity, Theorem 22 strong κ-universality. □

It follows from the Katětov's results that, under $\kappa = \kappa^{<\kappa}$, $\mathcal{U}_\kappa$ is isometric to his universal spaces. In fact, uniqueness follows from homogeneity using the original Urysohn procedure shown in the third section.

4.3. Questions

There are several questions related to the previous construction.

1. Katětov constructed a non-complete strongly ω-universal and ω-homogeneous space $\mathcal{W}$. Can one describe such a space using the space $\mathcal{U}_\omega$?
2. Katětov proved that there is no κ-universal and κ-homogeneous space having density κ if $\kappa < \kappa^{<\kappa}$. We believe that this result can be proved using $\mathcal{U}_\kappa$. One should prove that density of $\mathcal{U}_\kappa$ is bigger than κ in this case. One possibility might be to embed a discrete metric space X into $\mathcal{U}_\kappa$ (onto matrices having 1's outside diagonal) and for a given λ and sets $S \subset X$ of cardinalities λ take the distance matrices for quotients of X sewing S into one point. I think that κ^λ of such matrices will form a discrete space again.
3. V. Uspenskii used the Katětov's construction for $\kappa = \omega$ to universality of the group of isometries of Urysohn universal space. Is it possible to use the space $\mathcal{U}_\kappa$ for similar results?

Acknowledgement

The author would like to thank to a referee for valuable remarks and suggestions.

References

[1] M. Fréchet, Les dimensions d'unensemble abstrait, Math. Ann. 68 (1910) 145–168.
[2] M. Fréchet, L'expression la plus générale de la «distance» sur une droite, Amer. J. Math. 67 (1925) 1–10.
[3] F. Hausdorff, Gesammelte Werke, Band III (Deskriptive Mengenlehre und Topologie), Springer, Berlin, 2008.
[4] G.E. Huhunaišvili, On a property of Uryson's universal metric space, Dokl. Akad. Nauk SSSR 101 (1955) 607–610 (in Russian).
[5] M. Katětov, On universal metric spaces, in: Proc. 6th Prague Top. Symp. 1986, Heldermann, Berlin, 1988, pp. 323–330.
[6] J. Melleray, Some geometric and dynamical properties of the Urysohn space, in this issue.
[7] S. Mrówka, Solution d'un problème d'Urysohn concernant les espaces métriques universels, Bull. Acad. Polon. Sci. 1 (1953) 233–234.
[8] V.G. Pestov, Dynamics of Infinite-Dimensional Groups. The Ramsey–Dvoretzky–Milman Phenomenon, Univ. Lecture Ser., vol. 40, Amer. Math. Soc., 2006.
[9] V.M. Tikhomirov, Two letters to P.S. Uryson, Voprosy Istor. Estestvoznan. i Tekhn. 4 (1998) 66–69 (in Russian).
[10] P.S. Urysohn, Sur les espace métrique universel, C. R. Acad. Sci. Paris 180 (1925) 803–806.
[11] P.S. Uryson, Sur les espace métrique universel, Bull. Sci. Math. (1927) 43–64;
P.S. Uryson, Sur les espace métrique universel, Bull. Sci. Math. (1927) 74–90.
[12] P.S. Urysohn, in: P.S. Aleksandrov (Ed.), Trudy po Topologii i Drugim Oblastiam Matematiky, Gos. Izd. Tech.-Teor. Literatury, Moscow, 1951.
[13] V.V. Uspenskii, On the group of isometries of the Urysohn universal metric space, Comment. Math. Univ. Carolin. 31 (1990) 181–182.
[14] V.V. Uspenskii, The Urysohn universal metric space is homeomorphic to a Hilbert space, Topology Appl. 139 (2004) 145–149.
[15] V.V. Uspenskii, On subgroups of minimal topological groups, in this issue.